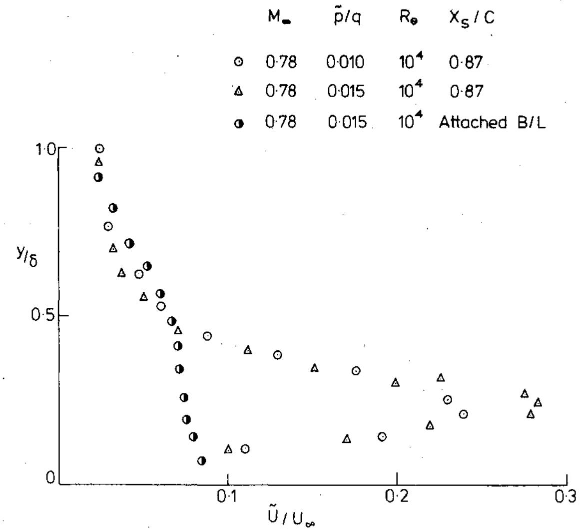

Fig. 4 Influence of tunnel noise on pressure fluctuation spectra.

layer at the same freestream Mach number and with $\tilde{p}/q_\infty \simeq 0.015$. The turbulence intensity levels in the separated layer indicate an amplification factor of 3-4 which is of the same order as that for the pressure fluctuation levels. It is observed that there are no significant changes in the turbulence levels with the change in the tunnel noise level from $\tilde{p}/q_\infty = 0.010$ to 0.015. The freestream level of turbulence is unaffected by separation and change in the tunnel noise.

The spectra of pressure fluctuation in the region of shock/boundary-layer interaction for a tunnel noise level $\tilde{p}/q_\infty = 0.01$ are shown in Fig. 3. The increase in pressure fluctuation levels in the separated region essentially occur at higher frequencies. The peak frequencies shift progressively across the shock. The peak frequency based upon the separation bubble length is $n_b = 2$. The peak frequencies based upon the chord length occur at $n \simeq 4$ with harmonics at $n \simeq 8$ and $\simeq 12$. The peak frequencies upstream and downstream of the interaction ($x/c = -0.75$ and 2.25) are also $n \simeq 4$. This suggests a correlation between the tunnel noise and the pressure fluctuation levels in the separated layer.

As shown in Fig. 4, changes in the wind-tunnel test section configuration to produce different noise levels have an effect on the pressure fluctuation spectra within the region of shock/boundary-layer interaction but do not effect the length of the separation bubble.

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References

[1] Raghunathan, S., Coll, J. B., and Mabey, D. G., "Flat Plate Turbulent Boundary Layers Subjected to Large Pressure Fluctuations," *AIAA Journal*, Vol. 17, Jan. 1979, pp. 105-107.

[2] Raghunathan, S., Coll, J. B., and Mabey, D. G., "Transonic Shock/Boundary Layer Interaction Subjected to Large Pressure Fluctuations," *AIAA Journal*, Vol. 17, Dec. 1979, pp. 1404-1406.

[3] McDevitt, J. B., Levy, L. L., Jr., and Deiwert, G. S., "Transonic Flow about a Thick Circular-Arc Airfoil," *AIAA Journal*, Vol. 14, May 1976, pp. 606-613.

[4] Mabey, D. G., "Analysis and Correlation of Data on Pressure Fluctuations in Separated Flow," *Journal of Aircraft*, Vol. 9, Sept. 1972, pp. 642-645.

[5] Plotkin, K. J., "Shockwave Oscillations Driven by Turbulent Boundary Layer Fluctuations," *AIAA Journal*, Vol. 13, Aug. 1975, pp. 1036-1040.

[6] Chen, C. P., Sajben, M., and Krontil, J. C., "Shockwave Oscillations in a Transonic Diffuser Flow," *AIAA Journal*, Vol. 17, Oct. 1979, pp. 1076-1083.

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Algorithm for Rapid Integration of Turbulence Model Equations on Parabolic Regions

David C. Wilcox*
DCW Industries, Inc., Studio City, Calif.

Introduction

THE advent of advanced engineering models of turbulence such as those described by Launder and Spalding[1] has been accompanied by a need for advanced numerical methods. For most of these advanced models, conventional numerical methods often are quite difficult to apply and sometimes do not work at all. For example, for a relatively simple flow such as fully developed channel flow explicit time-marching algorithms, relaxation schemes and the shooting method fail to yield a converged solution. The purpose of this Note is to cite yet another occasion where integration of an advanced set of turbulence-model equations has been unusually difficult and to describe the resolution of the problems encountered.

While developing a three-dimensional boundary-layer program using a standard parabolic marching scheme, the author has found computing time with the Wilcox-Rubesin[2] two-equation turbulence model to be very lengthy. The long computing time occurs because converged solutions are possible only when very small streamwise steps are taken. This observation is consistent with results obtained by Rastogi and Rodi[3] who have devised a three-dimensional boundary-layer program which uses the Jones-Launder[4] two-equation turbulence model. Rastogi and Rodi find that their initial stepsize Δs must be of the order of 1/100th of the local boundary-layer thickness δ and that ultimately Δs generally cannot exceed $\delta/2$. By contrast, a typical mixing-length computation for a three-dimensional wing begins with $\Delta s/\delta \sim 4$ and maintains a $\Delta s/\delta$ ratio in excess of unity. Such a severe stepsize limitation would increase computing time over that required for the mixing-length model by a factor of 10-100, depending upon the ultimate Reynolds number. Even if such run times were tolerable, the computer storage requirements for a real airplane wing would be prohibitive. Clearly an innovation is needed. The next section presents the necessary innovation.

Analysis

Model Equations

To illustrate the essence of the problems encountered while integrating turbulence-model equations, we begin by casting the turbulent energy equation in discretized form for a *two-dimensional* boundary layer. For the Wilcox-Rubesin model the differential equation for turbulent energy e is

$$u\frac{\partial e}{\partial s} + v\frac{\partial e}{\partial y} = \gamma^* \frac{(\partial u/\partial y)^2}{\omega} e - \beta^* \omega e + \frac{\partial}{\partial y}\left\{(\nu + \sigma^* \epsilon)\frac{\partial e}{\partial y}\right\} \qquad (1)$$

where s is arclength along the body, y the distance normal to the surface, u and v the velocity components in the s and y directions, and ω the turbulent dissipation rate. The quantities ν and ϵ denote kinematic molecular and eddy viscosity, respectively, while γ^*, β^*, and σ^* are closure coefficients whose values have been empirically found to be 1, 9/100, and

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*President. Associate Fellow AIAA.

½, respectively. The dissipation rate satisfies a similar equation; for present purposes it is necessary to focus only upon Eq. (1), and we thus omit the equation for ω.

Discretization

Proceeding now to discretization of Eq. (1) we assume, for simplicity, that the mesh points are equally spaced in both the s and y directions. Conventional, second-order accurate, discretization approximations for boundary-layer parabolic marching schemes are as follows:

$$u\frac{\partial e}{\partial s} \doteq \frac{u}{\Delta s}(3e_{m+1,n} - 4e_{m,n} + e_{m-1,n}) \tag{2}$$

$$\left(v - \sigma^*\frac{\partial \epsilon}{\partial y}\right)\frac{\partial e}{\partial y} \doteq \frac{v - \sigma^*\partial\epsilon/\partial y}{2\Delta y}(e_{m+1,n+1} - e_{m+1,n-1}) \tag{3}$$

$$\left\{\gamma^*\frac{(\partial u/\partial y)^2}{\omega} - \beta^*\omega\right\}e \doteq \left\{\gamma^*\frac{(\partial u/\partial y)^2}{\omega} - \beta^*\omega\right\}e_{m+1,n} \tag{4}$$

$$(\nu + \sigma^*\epsilon)\frac{\partial^2 e}{\partial y^2} = \frac{\nu + \sigma^*\epsilon}{(\Delta y)^2}(e_{m+1,n+1} - 2e_{m+1,n} + e_{m+1,n-1}) \tag{5}$$

where $e_{m,n}$ denotes the value of e at $s=s_m$ and $y=y_n$. Quantities without subscripts are assumed known during the (typically) iterative solution procedure. Substituting Eqs. (2-5) into Eq. (1) and regrouping terms leads to the conventional tridiagonal matrix system as follows:

$$A_n e_{m+1,n-1} + B_n e_{m+1,n} + C_n e_{m+1,n+1} = D_n \tag{6}$$

where A_n, B_n, C_n, and D_n are defined by

$$A_n = -\left\{\frac{v - \sigma^*\partial\epsilon/\partial y}{\Delta y} + \frac{\nu + \sigma^*\epsilon}{(\Delta y)^2}\right\} \tag{7}$$

$$B_n = 3\frac{u}{\Delta s} + 2\frac{\nu + \sigma^*\epsilon}{(\Delta y)^2} - \gamma^*\frac{(\partial u/\partial y)^2}{\omega} + \beta^*\omega \tag{8}$$

$$C_n = \frac{v - \sigma^*\partial\epsilon/\partial y}{\Delta y} - \frac{\nu + \sigma^*\epsilon}{(\Delta y)^2} \tag{9}$$

$$D_n = \frac{u}{\Delta s}\{4e_{m,n} - e_{m-1,n}\} \tag{10}$$

Lack of Diagonal Dominance

Now, in order to have a diagonally dominant system, the condition

$$B_n \geq -(A_n + C_n) \tag{11}$$

must be satisfied. Substituting Eqs. (7-9) into Eq. (11) yields the following condition:

$$3\frac{u}{\Delta s} - \gamma^*\frac{(\partial u/\partial y)^2}{\omega} + \beta^*\omega \geq 0 \tag{12}$$

When dissipation exceeds production, so that $\beta^*\omega > \gamma^*(\partial u/\partial y)^2/\omega$, Eq. (12) is satisfied so long as we march in the direction of flow (i.e., so long as u and Δs are of the same sign). The system is then said to be unconditionally stable. However, when production exceeds dissipation, we have the following limit on stepsize:

$$\Delta s \leq \frac{3u\omega}{\gamma^*(\partial u/\partial y)^2 - \beta^*\omega^2} \tag{13}$$

To test the validity of Eq. (13), results of an incompressible flat-plate boundary-layer (FPBL) computation have been analyzed. At a plate-length Reynolds number Re_s of 1.2×10^6 we have found empirically that stable computation can be achieved provided we do not exceed a Reynolds number based on Δs of 2.2×10^4, which corresponds to a ratio of Δs to boundary-layer thickness δ of 1.15. Figure 1 shows $Re_{\Delta s}$ as predicted by Eq. (13) throughout the boundary layer. As shown, the minimum value of $Re_{\Delta s}$ according to Eq. (13) is 1.9×10^4 and occurs at $y/\delta \doteq 0.012$. This close agreement shows that the source of instability is lack of diagonal dominance in the tridiagonal matrix system defined in Eqs. (6-10).

A New Algorithm

To remedy this situation, we note first that because of nonlinearity, Eq. (6) always requires an iterative solution. Letting superscript i denote iteration number, we replace B_n and D_n by the following discretization approximations:

$$B_n = 3\frac{u}{\Delta s} + 2\frac{\nu + \sigma^*\epsilon}{(\Delta y)^2} - \gamma^*\frac{(\partial u/\partial y)^2}{\omega} + (1+\psi_e)\beta^*\omega \tag{8a}$$

$$D_n = \frac{u}{\Delta s}\{4e_{m,n} - e_{m-1,n}\} + \psi_e\beta^*\omega e^{i-1}_{m+1,n} \tag{10a}$$

where ψ_e will be defined below. Then Eq. (6) is replaced by:

$$A_n e^i_{m+1,n-1} + B_n e^i_{m+1,n} + C_n e^i_{m+1,n+1} = D_n \tag{6a}$$

Inspection of Eqs. (6a), (8a) and (10a) shows that when convergence has been achieved (i.e., when $e^i_{m,n}$ and $e^{i-1}_{m,n}$ differ by a negligible amount), the terms on the right- and left-hand sides which are proportional to ψ_e cancel identically. Hence, the basic equation has in no way been modified. The advantage of this procedure becomes obvious upon inspection of the stability condition which now becomes:

$$\frac{3u}{\Delta s} \geq \frac{\gamma^*(\partial u/\partial y)^2}{\omega} - (1+\psi_e)\beta^*\omega \tag{12a}$$

Clearly, ψ_e can be chosen to insure that the right-hand side of Eq. (12a) is always negative which corresponds to *unconditional stability*, i.e., no limit on streamwise stepsize. Note that a quantity analogous to ψ_e must be defined (in a similar way) to insure stable integration of the turbulent dissipation rate equation.

Specification of ψ_e

Numerical experimentation has shown that the best results are obtained when $(1+\psi_e)\beta^*\omega$ exceeds $\gamma^*(\partial u/\partial y)^2/\omega$ by about 30%, a condition which is insured by defining ψ_e as follows:

$$\psi_e = \begin{cases} 3/10 & , \gamma^*(\partial u/\partial y)^2 \leq \beta^*\omega^2 \\ \dfrac{\gamma^*(\partial u/\partial y)^2}{\beta^*\omega^2} - \dfrac{7}{10} & , \gamma^*(\partial u/\partial y)^2 > \beta^*\omega^2 \end{cases} \tag{14}$$

It is interesting to note that numerical experimentation also shows that if ψ_e is too large, say $\psi_e = 2$, stable integration is inhibited. This phenomenon results from the nonlinear interaction between the streamwise convection term and the source terms in the ω equation. This can be shown by performing a linear stability analysis on the following limiting form of the Wilcox-Rubesin turbulent dissipation rate equation:

$$u\frac{\partial\omega^2}{\partial s} = \gamma\left(\frac{\partial u}{\partial y}\right)^2\omega - \beta\omega^3 \tag{15}$$

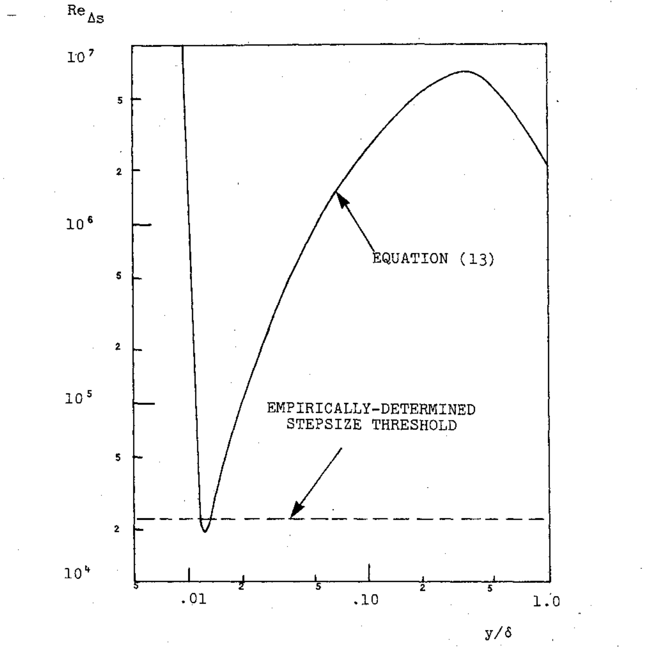

Fig. 1 Comparison of theoretical and empirically determined stepsize threshold for a flat-plate boundary layer, $Re_s = 1.2 \times 10^6$.

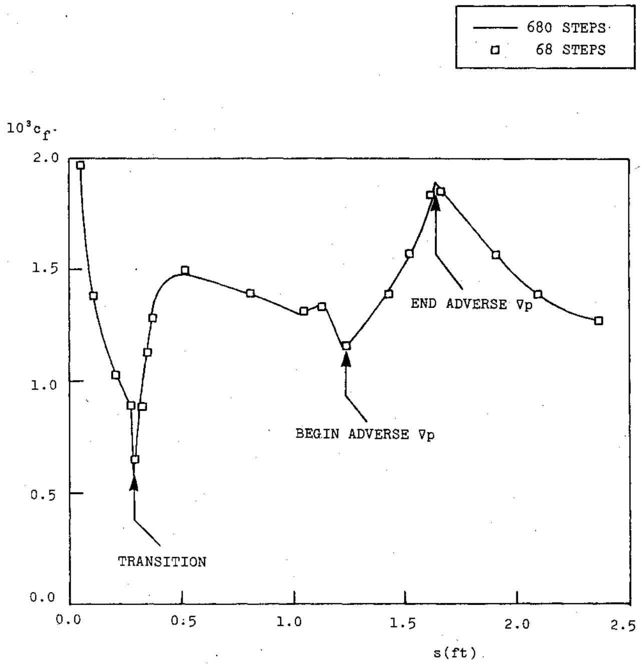

Fig. 2 Comparison of computed skin friction for Lewis-Gran-Kubota Mach 3 boundary layer with adverse and favorable pressure gradients.

where $\gamma = 1/3$ and $\beta = 3/20$. It is easy to show that solution errors will diminish from one iteration to the next provided the ω equation analog to ψ_e, viz, ψ_ω satisfies the following condition:

$$\psi_\omega < 1 + \frac{\gamma(\partial u/\partial y)^2}{\beta\omega^2} \tag{16}$$

which is consistent with results of the numerical experimentation noted above. In all of the applications presented below, the assumption has been made that the quantities ψ_e and ψ_ω are equal so that we write

$$\psi_\omega = \psi_e \tag{17}$$

Applications

Using the new algorithm, the FPBL computation has been repeated with larger Δs. With negligible sacrifice in accuracy, stable computation is possible up to an increase of a factor of 8. Stable computation is possible when Δs increases by a factor of 16, although numerical oscillations begin to develop because of insufficient resolution.

To test the new algorithm further, the Mach 3 compressible boundary-layer experimentally documented by Lewis et al.[5] has been computed first by using 680 steps in the streamwise direction and then by using 68 steps (so that Δs increases by a factor of 10). Figure 2 compares computed skin friction c_f for the two computations. As shown, computed c_f values are generally within 1-2% with a peak difference of about 5% near the sharp c_f peak at $s \doteq 3.7$ m (1.65 ft). Note that the factor of 10 increase in Δs causes no numerical problems either through transition [at $s \doteq 0.9$ m (0.3 ft)] or near the point of sudden application of adverse pressure gradient. Using the larger Δs reduces computing time for the Lewis-Gran-Kubota case by a factor of 4. The reason the reduction in computing time is less than a factor of 10 is because the number of iterations increases from one or two to an average of five or six.

Discussion

Based on these results, we have finally solved what has been a long-standing problem of limited stepsize for boundary-layer computations which use advanced turbulence-model equations. Generalization of the method is straightforward for a block tridiagonal system such as that used in many three-dimensional boundary-layer programs. The reduction in computing time attending the increase in maximum permissible Δs is as dramatic in the three-dimensional case as it is in the two-dimensional case.[6]

Acknowledgment

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References

[1]Launder, B. E. and Spalding, D. B., *Mathematical Models of Turbulence*, Academic Press, London, 1972.

[2]Wilcox, D. C. and Rubesin, M. W., "Progress in Turbulence Modeling for Complex Flowfields," NASA TP-1517, 1979.

[3]Rastogi, A. K. and Rodi, W., "Calculation of General Three-Dimensional Turbulent Boundary Layers," *AIAA Journal*, Vol. 16, Feb. 1978, pp. 151-159.

[4]Jones, W. P. and Launder, B. E., "Calculation of Low Reynolds Number Phenomena with a Two-Equation Model of Turbulence," *International Journal of Heat and Mass Transfer*, Vol. 16, 1973, pp. 1119-1129.

[5]Lewis, J. E., Gran, R. L., and Kubota, T., "An Experiment on the Adiabatic Compressible Turbulent Boundary Layer in Adverse and Favourable Pressure Gradients," *Journal of Fluid Mechanics*, Vol. 51, Pt. 4, 1972, pp. 657-672.

[6]Wilcox, D. C., "Computation of Three-Dimensional Boundary Layers with a Two-Equation Turbulence Model," DCW Industries, Inc., Studio City, Calif., Rept. DCW-R-11-03, June 1978.